Exercise 2.4.8

Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth $\dot{N} = -aN\ln(bN)$. (As in Exercise 2.3.3, $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.)

Solution

The fixed points occur where $\dot{N} = 0$.

$$-aN^*\ln(bN^*) = 0$$

$$-aN^* = 0 \quad \text{or} \quad bN^* = 1$$

$$N^* = 0 \quad \text{or} \quad N^* = \frac{1}{b}$$

Apply linear stability analysis to determine whether each of these points is stable or unstable.

$$f(N) = -aN\ln(bN)$$

Differentiate $f(N)$.

$$\begin{aligned} f'(N) &= -a\ln(bN) - a \\ &= -a[\ln(bN) + 1] \end{aligned}$$

As a result,

$$f'(0) = -a(-\infty) > 0 \qquad \Rightarrow \qquad N^* = 0 \text{ is an unstable fixed point.}$$

$$f'\left(\frac{1}{b}\right) = -a < 0 \qquad \Rightarrow \qquad N^* = 1/b \text{ is a stable fixed point.}$$

The graph of $(b/a)\dot{N}$ versus bN below confirms these results.